

Hairy black holes in the bigravity theory

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We study static, spherically symmetric black holes in the recently proposed massive gravity theory with two dynamical metrics that is maintained to be ghost-free. These solutions possess a regular event horizon common for both metrics, the values of the surface gravity and Hawking temperature calculated with respect to each metric being the same. The ratio of the event horizon radii measured by the two metrics is a free parameter that labels the solutions. We present a numerical evidence for their existence and find that they comprise several classes. Black holes within each class approach the same AdS-type asymptotic at infinity but differ from each other in the event horizon vicinity where the short-range massive modes reside. In addition, there are solutions showing a curvature singularity at a finite proper distance from the horizon. For some special solutions the graviton mass may become effectively imaginary, causing oscillations around the flat metric at infinity. The only asymptotically flat black hole we find – the Schwarzschild solution obtained by identifying the two metrics – seems to be exceptional, since changing even slightly its horizon boundary conditions completely changes the asymptotic behavior at infinity. We also construct globally regular solutions describing ‘lumps of pure gravity’ which show the same far field behavior as the black holes. However, adding a matter source gives asymptotically flat solutions exhibiting the Vainstein mechanism of recovery of General Relativity in a finite region.

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I. INTRODUCTION

A possibility that gravitons may have a mass [1] was thought for a long time to have mainly an academic interest. However, the recent observational evidence [2] suggests taking this idea more seriously, because it could provide an explanation for the current acceleration of our universe [3]. This has produced an increase of interest towards massive gravity theories (see [4] for a recent review).

Such theories are known to have serious theoretical difficulties – the absence of a smooth massless limit [5], presence of the ghost state in the spectrum [6], and the very low ultra-violet cutoff [7]. However, it seems that remedies may exist for some or perhaps for all of these problems. For example, the van-Dam-Veltman-Zakharov (vDVZ) discontinuity in the massless limit [5] seems to be curable by the Vainshtein mechanism [8], as was recently confirmed by the explicit calculation [9]. In addition, it seems that the presence of the ghost and the absence of uniqueness may cure each other, since among many possible theories of massive gravity there could be one that is ghost-free.

A special massive gravity model which is the only one that is ghost-free in the decoupling limit was recently discovered by de Rham, Gabadadze, and Tolley (RGT) [10]. A number of reports have then maintained that this theory should be completely free of the ghost [11], although the opposite claim has been advocated as well [12]. Setting aside this apparently difficult issue, one should say that the theory of [10], called below RGT model, is certainly distinguished. For example, some of its equations of motion ‘miraculously’ become algebraic, instead of being differential. This model has received a lot of attention, in particular its static solutions [13], [14] and time-dependent cosmological solutions [16] have been studied.

The RGT model is a bimetric theory containing a dynamical metric $g_{\mu\nu}$ and a non-dynamical flat reference metric $f_{\mu\nu}$. Quite recently, its bigravity generalization was proposed by Hassan and Rosen (HR), within which the metric $f_{\mu\nu}$ is also dynamical, so that the theory contains two gravitons – one massless and another one massive [17]. Moreover, it has been maintained that the generalized model remains ghost-free [18]. This theory contains the RGT model as a special limit, but it has richer dynamics, since it contains more degrees of freedom. For example, it reproduces General Relativity when the two metrics are identified, whereas otherwise the massive degrees of freedom are present. In this respect the theory resembles the standard gauge field theories, where the massive states are generated by assuming a

non-zero value for the Higgs field, but one can switch off the mass by setting the Higgs field to zero. Cosmological solutions in the HR theory were studied in [19], [20] while some static solutions were obtained in [21].

In the present paper we carry out a detailed analysis of static, spherically symmetric black holes in the HR theory of [17]. Our motivation is provided by the fact that the only known up to now black holes in massive gravity are all described by the Schwarzschild-(anti)de Sitter type metrics. These are exact solutions for which the metrics $g_{\mu\nu}$ and $f_{\mu\nu}$ are not simultaneously diagonal, they were discovered long ago in a particular massive gravity model [22], but later similar solutions were found in many other massive gravity theories [13], [14], [15]. It is, however, unclear if other, asymptotically flat, say, black holes can exist in massive gravity.

The two metrics can also be chosen to be simultaneously diagonal, but this leads to rather complicated equations. These equations admit asymptotically flat (numerical) solutions when a regular matter source is included [9], but it is unclear if their black hole solutions exist as well. The recent results seem to exclude such a possibility at least in the bimetric models where one of the metrics is non-dynamical and flat. Specifically, it turns out that when one numerically integrates the field equations starting from infinity towards an inner region, one does not find a regular event horizon but a naked singularity instead [23]. In addition, a geometric analysis of [24] shows that if a regular horizon is present, then it should be common for both metrics, which is impossible if one of them is flat.

At the same time, asymptotically flat black holes certainly exist in the bigravity theory, where both metrics are dynamical. These are the usual vacuum black holes obtained by setting $g_{\mu\nu} = f_{\mu\nu}$. It is then natural to try to find their generalizations to the case where $g_{\mu\nu} \neq f_{\mu\nu}$ and the massive degrees of freedom are excited. Such solutions could describe black holes surrounded by a cloud of massive hair. We therefore study in what follows black holes in the theory of [17], focusing in the main part of the paper on the case where both metrics are diagonal, the opposite case being considered in the Appendix.

We assume a regular event horizon and starting from it integrate numerically the field equations towards infinity. We find that the horizon is common for both metrics and that the values of the surface gravity and Hawking temperature calculated with respect to each metric are the same, which agrees with the conclusions of Ref.[24]. The ratio of the event horizon radii measured by the two metrics is a free parameter that labels the solutions,

so it is sufficient to vary only this parameter to construct all possible black holes. We then proceed to classify the solutions and find the following generic types of behavior: the solutions approach at infinity either the anti-de Sitter (AdS) metric, or special metrics that we call U, a backgrounds (see Eq.(4.7) below), or they develop a curvature singularity at a finite proper distance from the horizon. There are many solutions which run to the same AdS or U, a asymptotic at infinity but differ from each other in the event horizon vicinity, where a short massive ‘hair’ is localized. On the other hand, the asymptotically flat solution of Schwarzschild seems to be exceptional, since changing even slightly its horizon boundary conditions immediately changes the asymptotic behavior at infinity. As a result, we do find black holes with massive hair, but they are not asymptotically flat. It seems that asymptotic flatness is incompatible with the presence of a black hole horizon when the massive degrees of freedom are excited.

For the sake of completeness, we also briefly study solutions without a horizon which describe ‘lumps of pure gravity’. They have a regular center, but they are also not asymptotically flat and show in the far field the same behavior as the black holes. It seems they can be obtained from the latter in the limit where the horizon shrinks to zero. However, adding a matter source we do find asymptotically flat solutions which exhibit the Vainstein mechanism of recovery of General Relativity at finite distances. This suggests that the mechanism needs a matter source and so does not work for pure vacuum systems like black holes.

In the following three sections we describe the bigravity theory of [17], its equations of motion, the reduction to the spherically symmetric sector, and simple exact solutions that we call ‘background black holes’ and ‘ U, a backgrounds’. Sec.V explains our procedure of numerical integration of the equations starting from the horizon. In Sec.VI we study solutions which are parametrically close to the analytically known background black holes. The generic parameter values are considered in Sec.VII, with some limit cases analyzed in Sec.VIII. In Sec.IX we present globally regular solutions without a horizon, while Sec.X. contains concluding remarks. Finally, the Appendix describes exact black hole solutions with non-simultaneously diagonal metrics.

II. THE BIGRAVITY THEORY

The theory of [17] is defined on a four-dimensional spacetime manifold \mathcal{M} spanned by coordinates x^μ and equipped with two metrics $g_{\mu\nu}(x)$ and $f_{\mu\nu}(x)$. Their kinetic terms are chosen to be of the standard Einstein-Hilbert form with the couplings denoted by G and $\tan^2 \eta G$. We parameterize the action as

$$S = -\frac{1}{8\pi G} \int \left(\frac{1}{2} R + m^2 \cos^2 \eta \mathcal{L}_{\text{int}} \right) \sqrt{-g} d^4x - \frac{1}{16\pi \tan^2 \eta G} \int \mathcal{R} \sqrt{-f} d^4x + S_{(\text{m})}, \quad (2.1)$$

where R and \mathcal{R} are the Ricci scalars for $g_{\mu\nu}$ and $f_{\mu\nu}$, respectively, while S_{m} describes ordinary matter (for example perfect fluid) which is supposed to directly interact only with $g_{\mu\nu}$. The interaction between the two metrics is defined by

$$\mathcal{L}_{\text{int}} = \frac{1}{2} (K^2 - K^\nu_\mu K^\mu_\nu) + \frac{c_3}{3!} \epsilon_{\mu\nu\rho\sigma} \epsilon^{\alpha\beta\gamma\sigma} K^\mu_\alpha K^\nu_\beta K^\rho_\gamma + \frac{c_4}{4!} \epsilon_{\mu\nu\rho\sigma} \epsilon^{\alpha\beta\gamma\delta} K^\mu_\alpha K^\nu_\beta K^\rho_\gamma K^\sigma_\delta \quad (2.2)$$

with $K^\mu_\nu = \delta^\mu_\nu - \gamma^\mu_\nu$ and γ^μ_ν defined by the relation

$$\gamma^\mu_\sigma \gamma^\sigma_\nu = g^{\mu\sigma} f_{\sigma\nu} \quad (2.3)$$

where $g^{\mu\nu}$ is the inverse of $g_{\mu\nu}$. Apart from the gravitational coupling G the theory contains three parameters η, c_3, c_4 as well as the graviton mass m . In the limit where $\eta \rightarrow 0$ and $f_{\mu\nu}$ is flat the theory reduces to the RGT model of [10]. The above parametrization of the theory was also considered in Ref.[19] (with $\tan \eta \rightarrow \eta$, $m \cos \eta \rightarrow m$, $K^\mu_\nu \rightarrow -K^\mu_\nu$, $c_3 \rightarrow -c_3$).

In order to calculate γ^μ_σ defined by (2.3), it is convenient to introduce two tetrads e^μ_A and ω^A_μ defined by the conditions $g^{\mu\nu} = \eta^{AB} e^\mu_A e^\nu_B$ and $f_{\mu\nu} = \eta_{AB} \omega^A_\mu \omega^B_\nu$ with $\eta_{AB} = \text{diag}(1, -1, -1, -1)$. These tetrads are defined up to the local $SL(1, 3) \times SL(1, 3)$ rotations, which freedom can be used to impose the conditions

$$e^\mu_A \omega_{B\mu} = e^\mu_B \omega_{A\mu} \quad (2.4)$$

with $\omega_{A\mu} = \eta_{AB} \omega^B_\mu$, which insures that

$$\gamma^\mu_\nu = e^\mu_A \omega^A_\nu. \quad (2.5)$$

Varying the action gives the field equations (see [19] for details)

$$G^\rho_\lambda = m^2 \cos^2 \eta T^\rho_\lambda + 8\pi G T^{(\text{m})\rho}_\lambda, \quad (2.6)$$

$$\mathcal{G}^\rho_\lambda = m^2 \sin^2 \eta \mathcal{T}^\rho_\lambda, \quad (2.7)$$

where G_λ^ρ and \mathcal{G}_λ^ρ are the Einstein tensors for $g_{\mu\nu}$ and $f_{\mu\nu}$, respectively, while

$$T_\lambda^\rho = \tau_\lambda^\rho - \delta_\lambda^\rho \mathcal{L}_{\text{int}}, \quad \mathcal{T}_\lambda^\rho = -\frac{\sqrt{-g}}{\sqrt{-f}} \tau_\lambda^\rho, \quad (2.8)$$

with

$$\tau_\lambda^\rho = (\gamma_\sigma^\sigma - 3)\gamma_\lambda^\rho - \gamma_\sigma^\rho \gamma_\lambda^\sigma + \frac{c_3}{2} \epsilon_{\lambda\mu\nu\sigma} \epsilon^{\alpha\beta\gamma\sigma} \gamma_\alpha^\rho K_\beta^\mu K_\gamma^\nu + \frac{c_4}{6} \epsilon_{\lambda\mu\nu\sigma} \epsilon^{\alpha\beta\gamma\delta} \gamma_\alpha^\rho K_\beta^\mu K_\gamma^\nu K_\delta^\sigma. \quad (2.9)$$

The Bianchi identities for the left-hand side of Eq.(2.6) imply the conservation condition

$$\overset{(g)}{\nabla}_\rho T_\lambda^\rho = 0, \quad (2.10)$$

where $\overset{(g)}{\nabla}_\rho$ is the covariant derivative with respect to $g_{\mu\nu}$. In view of the diffeomorphism-invariance of the interaction term $S_{\text{int}} = \int \mathcal{L}_{\text{int}} \sqrt{-g} d^4x$, the conditions $\overset{(f)}{\nabla}_\rho \mathcal{T}_\lambda^\rho = 0$ then follows identically. The matter energy-momentum tensor is conserved independently, in view of the diffeomorphism-invariance of the matter action $S_{(\text{m})}$,

$$\overset{(g)}{\nabla}_\rho T^{(\text{m})\rho}_\lambda = 0. \quad (2.11)$$

If the matter source vanishes, then choosing the two metrics to be the same reduces the field equations to those of vacuum General Relativity,

$$g_{\mu\nu} = f_{\mu\nu}, \quad T^{(\text{m})\rho}_\lambda = 0 \quad \Rightarrow \quad G_\lambda^\rho = \mathcal{G}_\lambda^\rho = 0, \quad (2.12)$$

since one has in this case $T_\nu^\mu = \mathcal{T}_\nu^\mu = 0$.

In what follows we shall be considering solutions of equations (2.6), (2.7) within the spherically symmetric sector. Introducing the spherical coordinates $x^\mu = (t, r, \vartheta, \varphi)$, the most general expressions for the two tetrads are [19]

$$\begin{aligned} e_0 &= \frac{1}{Q} \frac{\partial}{\partial t}, & e_1 &= N \frac{\partial}{\partial r}, & e_2 &= \frac{1}{R} \frac{\partial}{\partial \vartheta}, & e_3 &= \frac{1}{R \sin \vartheta} \frac{\partial}{\partial \varphi}, \\ \omega^0 &= a dt + c dr, & \omega^1 &= -c Q N dt + b dr, & \omega^2 &= U d\vartheta, & \omega^3 &= U \sin \vartheta d\varphi, \end{aligned} \quad (2.13)$$

where Q, N, R, a, b, c, U are functions of r . The corresponding metrics read

$$g_{\mu\nu} dx^\mu dx^\nu = Q^2 dt^2 - \frac{dr^2}{N^2} - R^2 (d\vartheta^2 + \sin^2 \vartheta d\varphi^2) \quad (2.14)$$

and

$$f_{\mu\nu} dx^\mu dx^\nu = (a^2 - c^2 Q^2 N^2) dt^2 + 2c(a + bQN) dt dr - (b^2 - c^2) dr^2 - U^2 (d\vartheta^2 + \sin^2 \vartheta d\varphi^2), \quad (2.15)$$

while

$$\gamma^\mu_\nu = e^\mu_A \omega^A_\nu = \begin{pmatrix} a/Q & c/Q & 0 & 0 \\ -cQN^2 & bN & 0 & 0 \\ 0 & 0 & U/R & 0 \\ 0 & 0 & 0 & U/R \end{pmatrix}. \quad (2.16)$$

It is now straightforward to compute \mathcal{L}_{int} and the tensor τ^μ_ν defined by (2.9) (the explicit expressions are given in the Appendix in [19]).

It is still possible to reparameterize the radial coordinate to impose the gauge condition $R(r) = r$ so that the metric $g_{\mu\nu}$ becomes

$$g_{\mu\nu} dx^\mu dx^\nu = Q^2 dt^2 - \frac{dr^2}{N^2} - r^2 (d\vartheta^2 + \sin^2 \vartheta d\varphi^2). \quad (2.17)$$

Since its Einstein tensor is diagonal, so should be the energy-momentum tensor. Therefore, one has to have $T_r^0 = 0$, which requires that $\tau_r^0 = 0$. Using (2.9) one finds

$$\tau_r^0 = \frac{c}{Qr^2} (r(2U - 3r) + c_3(3r - U)(r - U) + c_4(r - U)^2), \quad (2.18)$$

and for this to vanish, one should either have $c = 0$, or choose $c \neq 0$ but set to zero the expression between the parenthesis. The latter case is studied in the Appendix, where it is shown that the metric $g_{\mu\nu}$ will be described in this situation by the Schwarzschild-(anti)de Sitter solution. More general metrics are obtained by choosing $c = 0$.

III. FIELD EQUATIONS

If $c = 0$ then the metric $f_{\mu\nu}$ in (2.15) becomes diagonal. Assuming that all its coefficient depend only on r and introducing a new function Y via $b = U'/Y$ with $' \equiv d/dr$ the metric reads

$$f_{\mu\nu} dx^\mu dx^\nu = a^2 dt^2 - \frac{U'^2}{Y^2} dr^2 - U^2 (d\vartheta^2 + \sin^2 \vartheta d\varphi^2). \quad (3.1)$$

We thus have two static and spherically symmetric metrics (2.17) and (3.1) which contain 5 functions of r : Q , N , a , Y , U . The Einstein equations (2.6),(2.7) reduce then to the following system,

$$\begin{aligned} G_0^0 &= m^2 \cos^2 \eta T_0^0 + \rho, \\ G_r^r &= m^2 \cos^2 \eta T_r^r - P, \\ \mathcal{G}_0^0 &= m^2 \sin^2 \eta \mathcal{T}_0^0, \\ \mathcal{G}_r^r &= m^2 \sin^2 \eta \mathcal{T}_r^r, \end{aligned} \quad (3.2)$$

which should be supplemented by the conservation condition $\overset{(g)}{\nabla}_\rho T_\lambda^\rho = 0$,

$$(T_r^r)' + \frac{Q'}{Q} (T_r^r - T_0^0) + \frac{2}{r} (T_\vartheta^\vartheta - T_r^r) = 0. \quad (3.3)$$

As was said above, it is not necessary to require in addition that $\overset{(f)}{\nabla}_\rho \mathcal{T}_\lambda^\rho = 0$, since this condition follows from (3.2),(3.3). It is assumed in the above equations that the matter source is of the perfect fluid type,

$$8\pi G T_\lambda^{(m)\rho} = \text{diag}[\rho(r), -P(r), -P(r), -P(r)] \quad (3.4)$$

whose conservation requires that

$$P' = -\frac{Q'}{Q} (\rho + P). \quad (3.5)$$

The 5 equations (3.2),(3.3) explicitly read

$$\frac{2NN'}{r} + \frac{N^2 - 1}{r^2} + m^2 \cos^2 \eta \left(\alpha_1 \frac{N}{Y} U' + \alpha_2 \right) + \rho = 0, \quad (3.6a)$$

$$\frac{2N^2 Q'}{Qr} + \frac{N^2 - 1}{r^2} + m^2 \cos^2 \eta \left(\alpha_1 \frac{a}{Q} + \alpha_2 \right) - P = 0, \quad (3.6b)$$

$$\{Y^2 - 1 + m^2 \sin^2 \eta \alpha_3\} NU' + 2UNYY' + m^2 \sin^2 \eta Y \alpha_4 = 0, \quad (3.6c)$$

$$\{a(Y^2 - 1) + m^2 \sin^2 \eta \alpha_5\} U' + 2UY^2 a' = 0, \quad (3.6d)$$

$$\alpha_6 U' + \alpha_7 a' = 0, \quad (3.6e)$$

where the following abbreviations have been introduced,

$$\begin{aligned} \alpha_1 &= 3 - 3c_3 - c_4 + \frac{2(c_4 + 2c_3 - 1)U}{r} - \frac{(c_4 + c_3)U^2}{r^2}, \\ \alpha_2 &= 4c_3 + c_4 - 6 + \frac{2(3 - c_4 - 3c_3)U}{r} + \frac{(c_4 + 2c_3 - 1)U^2}{r^2}, \\ \alpha_3 &= c_4 U^2 - 2(c_3 + c_4)rU + (c_4 + 2c_3 - 1)r^2, \\ \alpha_4 &= (3 - c_4 - 3c_3)r^2 - (c_4 + c_3)U^2 + (4c_3 + 2c_4 - 2)rU, \\ \alpha_5 &= [(a - Q)c_4 - Qc_3]U^2 + [2(2Q - a)c_3 + (Q - a)c_4 - Q]rU, \\ &\quad + [(2a - 3Q)c_3 + (a - Q)c_4 + 3Q - a]r^2, \\ \alpha_6 &= Q'N[(3c_3 + c_4 - 3)r^2 + (2(1 - c_4 - 2c_3))Ur + (c_4 + c_3)U^2], \\ &\quad + 2Q(Y - N)[(3 - c_4 - 3c_3)r + (c_4 + 2c_3 - 1)U], \\ &\quad + 2a(N - Y)[(1 - c_4 - 2c_3)r + (c_4 + c_3)U], \\ \alpha_7 &= Y[(3 - c_4 - 3c_3)r^2 + 2(c_4 + 2c_3 - 1)Ur - (c_4 + c_3)U^2]. \end{aligned} \quad (3.7)$$

Eliminating the term Q' in α_6 with the use of (3.6b), the coefficients $\alpha_1, \dots, \alpha_7$ depend only on Q, N, a, Y, U but not on their derivatives.

The equations admit the scale symmetry that maps solutions to solutions,

$$\begin{aligned} N(r) &\rightarrow N(\lambda r), & Y(r) &\rightarrow Y(\lambda r), & U(r) &\rightarrow \frac{1}{\lambda} U(\lambda r), & Q(r) &\rightarrow Q(\lambda r), \\ a(r) &\rightarrow a(\lambda r), & m &\rightarrow \frac{m}{\lambda}, & \eta &\rightarrow \eta, & c_3 &\rightarrow c_3, & c_4 &\rightarrow c_4. \end{aligned} \quad (3.8)$$

In the following few sections, until Section IX, we shall set $\rho = P = 0$.

IV. SIMPLEST SOLUTIONS

Some exact solutions of equations (3.6a)–(3.6e) can be obtained.

A. Background black holes

Let us choose the two metrics to be conformally related,

$$f_{\mu\nu} = C^2 g_{\mu\nu}, \quad (4.1)$$

with constant C . Equations (3.6a)–(3.6e) will be fulfilled if C satisfies the algebraic equation

$$(C - 1)P(C) = 0, \quad (4.2)$$

with

$$P(C) = (c_3 + c_4)C^2 + (3 - 5c_3 + (\xi - 2)c_4)C + (4 - 3\xi)c_3 + (1 - 2\xi)c_4 - 6 + \frac{\xi(3c_3 + c_4 - 3)}{C}, \quad (4.3)$$

(here $\xi = \tan^2 \eta$), while $g_{\mu\nu}$ is the Schwarzschild-(anti)de Sitter metric, so that

$$Y^2 = N^2 = 1 - \frac{2M}{r} - \frac{\Lambda(C)}{3} r^2, \quad U = Cr, \quad a = CQ, \quad Q = qN, \quad (4.4)$$

where q is an arbitrary constant related to the time scaling symmetry, while

$$\Lambda(C) = m^2 \cos^2 \eta (1 - C) \{ (c_3 + c_4)C^2 + (3 - 5c_3 - 2c_4)C + 4c_3 + c_4 - 6 \}. \quad (4.5)$$

One root of Eq.(4.2) is $C = 1$, in which case $\Lambda = 0$ and we obtain the Schwarzschild solution. Depending on values of the parameters c_3, c_4, η , equation (4.2) can have up to three more

real roots, for which $\Lambda(C)$ does not generically vanish. For example, for $m = 0.1$, $\eta = 1$, $c_3 = 0.1$, $c_4 = 0.3$ Eq.(4.2) has altogether four real roots, $C = \{C_k\}$,

$$\begin{aligned} \{C_1, C_2, C_3, C_4\} &= \{1; -0.6458; 2.6333; -8.5566\}, \\ \frac{\Lambda(C_k)}{m^2} &= \{0; -3.0559; -1.1812; +21.5625\}. \end{aligned} \quad (4.6)$$

This gives the Schwarzschild (S) solution for $C = C_1$, Schwarzschild-anti-de Sitter (SAdS) solutions for $C = C_2, C_3$, and the Schwarzschild-de Sitter (SdS) solution for $C = C_4$. We shall call these solutions background black holes, since below they will be considered as reference backgrounds – the starting point for studying more general solutions.

It is worth noting that these solutions are not the same as those in the Appendix, where $g_{\mu\nu}$ is also SdS or SAdS but $g_{\mu\nu}$ and $f_{\mu\nu}$ are not proportional.

B. U, a backgrounds

Another class of solutions can be obtained by setting U, a to constant values:

$$\begin{aligned} N^2 &= 1 + m^2 \cos^2 \eta \left((1 - 2c_3 - c_4)U^2 - \frac{2M}{r} + (3c_3 + c_4 - 3)Ur + \left(2 - \frac{4}{3}c_3 - \frac{1}{3}c_4\right)r^2 \right), \\ \frac{Q}{N} &= a \frac{m^2 \cos^2 \eta}{2} \int_{r_1}^r \frac{dr}{xN^3} \mathcal{F}, \quad Y = \frac{m^2 \sin^2 \eta}{2U} \int_{r_2}^r \frac{dr}{N} \mathcal{F}, \end{aligned} \quad (4.7)$$

where $\mathcal{F} = (c_4 - 3 + 3c_3)x^2 + 2(1 - 2c_3 - c_4)Ux + (c_3 + c_4)U^2$ and M, r_1, r_2 are integration constants. For $r \rightarrow \infty$ one has $Q^2 \sim N^2 \sim Y \sim r^2$ and the metric $g_{\mu\nu}$ approaches in the leading order the (anti)de Sitter metric, but the subleading terms are different. The metric $f_{\mu\nu}$ is actually degenerate, since $f_{rr} = U'^2/Y^2 = 0$. However, such solutions will describe below the asymptotic behavior of other, more general solutions for which U, a become constant only for $r \rightarrow \infty$ so that f_{rr} vanishes only asymptotically. The proper distance till infinity $\int_r^\infty (U'/Y)dr$ and the proper volume of the 3-space are then finite, so that with respect to $f_{\mu\nu}$ the spacetime is spontaneously compactified.

V. BOUNDARY CONDITIONS AT THE HORIZON

Since we could not find other solutions of equations (3.6a)–(3.6e) in a closed analytic form, we wish to integrate these equations numerically. As a first step, one should resolve

the equations with respect to the derivatives, and our procedure is as follows. First of all, taking the ratio of Eqs.(3.6d) and (3.6e) leads to the algebraic relation

$$\frac{a(Y^2 - 1) + m^2 \sin^2 \eta \alpha_5}{\alpha_6} = \frac{2UY^2}{\alpha_7}, \quad (5.1)$$

which can be expressed in the form

$$\frac{a}{Q} = \mathcal{F}(r, N, Y, U, m, \eta, c_3, c_4). \quad (5.2)$$

Using this, Eqs.(3.6a), (3.6b), (3.6c) and (3.6e) reduce to

$$N' = \mathcal{F}_1 U' + \mathcal{F}_2, \quad (5.3a)$$

$$Y' = \mathcal{F}_3 U' + \mathcal{F}_4, \quad (5.3b)$$

$$Q' = \mathcal{F}_5 Q, \quad (5.3c)$$

$$a' = \mathcal{F}_6 Q U', \quad (5.3d)$$

where $\mathcal{F}_k = \mathcal{F}_k(r, N, Y, U, m, \eta, c_3, c_4)$. Eqs.(5.2), (5.3c),(5.3d) together imply that

$$\mathcal{F}' = \mathcal{F}_6 U' - \mathcal{F}\mathcal{F}_5, \quad (5.4)$$

and calculating the derivative on the left gives

$$\partial_r \mathcal{F} + \partial_N \mathcal{F}(\mathcal{F}_1 U' + \mathcal{F}_2) + \partial_Y \mathcal{F}(\mathcal{F}_3 U' + \mathcal{F}_4) + \partial_U \mathcal{F} U' = \mathcal{F}_6 U' - \mathcal{F}\mathcal{F}_5, \quad (5.5)$$

which can be resolved with respect to U' ,

$$U' = \frac{\partial_r \mathcal{F} + \partial_N \mathcal{F}\mathcal{F}_2 + \partial_Y \mathcal{F}\mathcal{F}_4 + \mathcal{F}\mathcal{F}_5}{\mathcal{F}_6 - \mathcal{F}_1 \partial_N \mathcal{F} - \mathcal{F}_3 \partial_Y \mathcal{F} - \partial_U \mathcal{F}} \equiv \mathcal{D}U. \quad (5.6)$$

Injecting this into (5.3a) and (5.3b) finally yields a system of 3 coupled equations,

$$\begin{aligned} N' &= \mathcal{D}N(r, N, Y, U, m, \eta, c_3, c_4), \\ Y' &= \mathcal{D}Y(r, N, Y, U, m, \eta, c_3, c_4), \\ U' &= \mathcal{D}U(r, N, Y, U, m, \eta, c_3, c_4), \end{aligned} \quad (5.7)$$

where the explicit expressions of the functions on the right are somewhat lengthy, so that we do not write them down explicitly. When a solution of these equations is found, then Q is obtained by integrating equation (5.3c), and finally a is obtained from (5.2).

We wish to study black hole solutions of Eqs.(5.7) and so assume that there is an event horizon at $r = r_h$ where $Q(r_h) = N(r_h) = 0$. For the horizon to be non-singular and non-degenerate, Q^2 and N^2 should both have simple zeros at $r = r_h$. Next, the analysis of Eqs.(5.7) shows that if N^2 has a simple zero at $r = r_h$ then Y^2 should have a simple zero at this point too. At the same time, U can assume at the horizon any finite value. We therefore assume the power-series expansions near $r = r_h$,

$$N^2 = \sum_{n \geq 1} a_n (r - r_h)^n, \quad Y^2 = \sum_{n \geq 1} b_n (r - r_h)^n, \quad U = ur_h + \sum_{n \geq 1} c_n (r - r_h)^n. \quad (5.8)$$

Injecting this to (5.7), all coefficients a_n, b_n, c_n can be expressed in terms of u, a_1 where u is arbitrary while a_1 satisfies $\mathcal{A} a_1^2 + \mathcal{B} a_1 + \mathcal{C} = 0$, where $\mathcal{A}, \mathcal{B}, \mathcal{C}$ are functions of $u, r_h, m, \eta, c_1, c_2$. There are two solutions for a_1 ,

$$a_1 = a_1^\pm(u) = \frac{1}{2\mathcal{A}} (-\mathcal{B} \pm \sqrt{\mathcal{B}^2 - 4\mathcal{A}\mathcal{C}}), \quad (5.9)$$

which give rise to two different local solutions (5.8) with different values of $(N^2)', (Y^2)', U'$ at $r = r_h$. Using these solutions in Eqs.(5.2),(5.3c) gives

$$Q^2 = q^2 \{r - r_h + \sum_{n \geq 2} c_n (r - r_h)^n\}, \quad a^2 = q^2 \sum_{n \geq 1} d_n (r - r_h)^n, \quad (5.10)$$

where c_n and d_n are expressed in terms of a_1 and u , while q is an integration constant that reflects the possibility to rescale the time coordinate for both metric simultaneously.

At this point it is interesting to compare our results with the geometric analysis of bimetric theories of Ref.[24]. Let us consider $\xi = \partial/\partial t$, the timelike Killing vector for the both metrics. Its norms calculated with respect to each metric, $\langle \xi, \xi \rangle_g = Q^2$ and $\langle \xi, \xi \rangle_f = a^2$, both vanish at $r = r_h$ so that the horizon is the Killing horizon for each of the two metrics at the same time. The surface gravity for each metric is the horizon value of

$$\begin{aligned} \kappa_g^2 &= -\frac{1}{2} g^{\mu\alpha} g_{\nu\beta} \overset{(g)}{\nabla}_\mu \xi^\nu \overset{(g)}{\nabla}_\alpha \xi^\beta = \lim_{r \rightarrow r_h} Q^2 N'^2 = \frac{1}{4} q^2 a_1, \\ \kappa_f^2 &= -\frac{1}{2} f^{\mu\alpha} f_{\nu\beta} \overset{(f)}{\nabla}_\mu \xi^\nu \overset{(f)}{\nabla}_\alpha \xi^\beta = \lim_{r \rightarrow r_h} a^2 \left(\frac{Y}{U'} \right)^2 = \frac{1}{4} q^2 \frac{d_1 b_1}{(c_1)^2}. \end{aligned} \quad (5.11)$$

The Hawking temperature for each metric is obtained in the standard way by passing to the imaginary time and requiring the absence of conical singularity. This gives

$$T_g = \frac{\kappa_g}{2\pi}, \quad T_f = \frac{\kappa_f}{2\pi}. \quad (5.12)$$

Now, using the explicit expressions for the coefficients a_1, b_1, c_1, d_1 (we do not write them down in view of their complexity) the ratio of the two surface gravities evaluates to one,

$$\frac{\kappa_g^2}{\kappa_f^2} = \frac{a_1(c_1)^2}{d_1 b_1} = 1, \quad (5.13)$$

for any u and for both signs in (5.9). Therefore, the surface gravities and the Hawking temperatures are the same. All this agrees with the conclusions of Ref.[24] that the two Killing horizons and their surface gravities should coincide.

Returning to our analysis, the black holes solutions are obtained by numerically extending the local solutions (5.8), (5.10) towards large r . It follows that the black holes are determined by the value of $u = U(r_h)/r_h$ – the ratio of the event horizon radius measured by $f_{\mu\nu}$ to that measured by $g_{\mu\nu}$ – as well as by the choice of sign in (5.9). Depending on the latter, we shall say that the solutions belong either to the upper or to the lower branch. Under the scale transformations (3.8) the ratio $u = U(r_h)/r_h$ stays invariant while $r_h \rightarrow r_h/\lambda$, and this can be used to set $r_h = 1$, which condition will be assumed from now on.

It is then relatively easy to explore the structure of the solution space, since it is enough to integrate the equations for various values of u , separately for each branch. The integration constant q in (5.10) is not an essential parameter and can be fixed afterwards, when the global solutions are already known, for example by requiring that $Q^2/N^2 \rightarrow 1$ as $r \rightarrow \infty$. Let us choose some values of the theory parameters, for example the same as in Eq.(4.6): $m = 0.1, \eta = 1, c_3 = 0.1, c_4 = 0.3$. There is nothing special about these values, since varying them changes the solutions smoothly, without changing their qualitative structure. We then use the numerical routines of [25] to integrate the equations starting from $r = r_h = 1$ towards large r .

To begin with, for $u = C$ with $C = \{C_k\}$ given by Eq.(4.6), the numerical solutions should reproduce the background black holes, for which $U'(r) = C$. Indeed, for the four values of C in Eq.(4.6) we find numerical solutions with $U'(r) = C$ and N, Q, Y coinciding with those in Eq.(4.4). The solutions for $C = \{C_1, C_2\}$ belong to the upper branch, while those for $C = \{C_3, C_4\}$ belong to the lower branch.

VI. HAIRY BLACK HOLES

As a next step, we choose $u = C + \delta u$ where δu is small. It is then natural to expect the solution to be a slightly deformed background black hole, the deformations being due to the ‘massive hair’ present for generic values of u . In what follows we analyze these expectations and find that the deformations are indeed small, but in general only within a finite neighborhood of the event horizon and not necessarily for all values of r .

A. Deformations of the asymptotically flat black hole.

The only asymptotically flat solution among the background black holes is the Schwarzschild solution (4.4) with $C = C_1 = 1$. We consider its deformed version for $u = 1 + \delta u$. It turns out that choosing δu to be negative does not give anything, since the local solutions (5.8) become then complex-valued. However, for δu positive and small enough, for example $\delta u = 10^{-2}$, the deformed solution exists and stays very close to the Schwarzschild solution in a large vicinity of the horizon, for $r < r_{\max}(u)$. However, for larger values of r it completely changes its structure, since the Q , N , Y amplitudes then grow rapidly, while a , U approach finite asymptotic values (see Fig.1), so that the whole configuration approaches one of the U, a backgrounds (4.7).

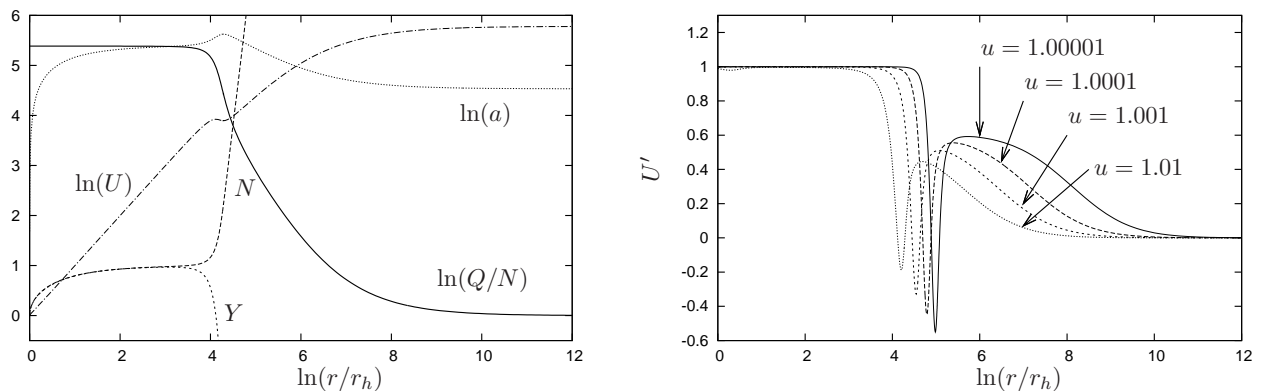


Figure 1. Left: solution profiles for $u = 1.01$. Right: $U'(r)$ for several values of u . Here and in Figs.2,3,4 below one has $m = 0.1$, $\eta = 1$, $c_3 = 0.1$, $c_4 = 0.3$.

It is instructive to consider the function $U'(r)$, which is equal to one everywhere for $u = 1$. For $u > 1$ it stays very close to one for $r < r_{\max}(u)$ but at $r \sim r_{\max}(u)$ it suddenly drops down

and after a couple of oscillations tends to zero at infinity. The value $r_{\max}(u)$ increases when u decreases, and in the limit $u \rightarrow 1$ one has $r_{\max}(u) \rightarrow \infty$ so that the solution approaches the Schwarzschild metric for any finite r . However, since the boundary conditions at infinity for $u > 1$ are not the same as for $u = 1$, the convergence in the limit $u \rightarrow 1$ is only pointwise and not uniform.

The conclusion is that exciting the massive degrees of freedom around the Schwarzschild black hole produces deformations which stay small close to the horizon but inevitably become large at infinity, thus destroying the asymptotic flatness.

The above conclusion is supported by the following counting argument. Let us *require* the solution to be asymptotically flat. Then for $r \rightarrow \infty$ one should have $N = 1 + \delta N$, $Y = 1 + \delta Y$, $U = r + \delta U$ where the variations are small. Inserting this into Eqs.(5.7) and linearizing with respect to δN , δY , δU and keeping only vanishing at infinity modes gives

$$\begin{aligned} N &= 1 - \frac{A \sin^2 \eta}{r} + B \cos^2 \eta \frac{mr + 1}{r} e^{-mr}, & U &= r + B \frac{m^2 r^2 + mr + 1}{m^2 r^2} e^{-mr}, \\ Y &= 1 - \frac{A \sin^2 \eta}{r} - B \sin^2 \eta \frac{1 + mr}{r} e^{-mr}, \end{aligned} \quad (6.1)$$

where A, B are integration constants. The other two metric amplitudes then read

$$Q = 1 - \frac{A \sin^2 \eta}{r} + \frac{2B \cos^2 \eta}{r} e^{-mr}, \quad a = 1 - \frac{A \sin^2 \eta}{r} - \frac{2B \sin^2 \eta}{r} e^{-mr}. \quad (6.2)$$

This asymptotic solution is the superposition of the long-range massless Newtonian mode and the short-range massive VdVZ mode [5].

Suppose that one wants to find black hole solutions with this asymptotic behavior using the multiple shooting method [25]. In this method one integrates the equations starting from the horizon towards large r , and at the same time starting from infinity towards small r . The two solutions should match at some intermediate point, which gives three matching conditions for N, Y, U . The matching conditions should be fulfilled by adjusting the free parameters, but since there are only two parameters A, B in (6.1), one cannot fulfill all three conditions. If one adjusts also the parameter u at the horizon, then it could be possible to construct global solutions, but these will exist at most for discrete sets of values of A, B, u . As a result, one cannot vary u continuously and so there could be no asymptotically flat continuous hairy deformations of the Schwarzschild solution.

The above argument does not exclude the U, a asymptotics (4.7), since they contain altogether 5 free parameters, which is enough to fulfill the matching conditions.

B. Hairy deformations of the asymptotically AdS black holes.

Let us now choose in (5.8) $u = C + \delta u$ where $C = C_2$ or $C = C_3$ defined by (4.6). This corresponds to deformations of the asymptotically AdS black holes with $\Lambda(C_2) = -3.0559 m^2$ or $\Lambda(C_3) = -1.1812 m^2$. Integrating the equations shows that solutions with such boundary conditions exist for both values of C , provided that $|\delta u|$ is not too large. For $r \rightarrow \infty$ the solutions approach the corresponding background black hole configuration (4.4), but they deviate from it close to the horizon. The solution profiles for $C = C_3 = 2.6333$ are shown in Fig.2, while those for $C = C_2$ look qualitatively similar. Since the deformations do not change the asymptotic behavior at infinity in this case, they can be viewed as short massive hair localized in the horizon vicinity.

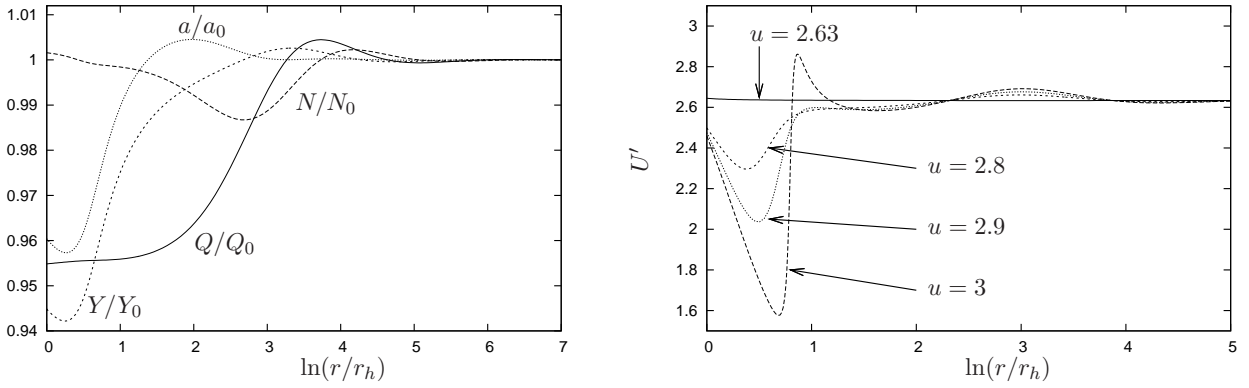


Figure 2. Left: profiles of the asymptotically AdS solution for $u = 2.8$, where N_0, Q_0, Y_0, a_0 correspond to the background black hole (4.4) with $C = C_3$. Right: $U'(r)$ for several values of u .

It is worth noting that, unlike for asymptotically flat solutions, a similar counting argument does not forbid the existence of continuous deformations of the asymptotically AdS black holes. In order to simplify the discussion, let us set $c_3 = c_4 = 0$, in which case Eq.(4.2) can be solved analytically,

$$C = 1 \pm \frac{1}{\cos \eta} \quad \Rightarrow \quad \Lambda(C) = m^2(\pm \cos \eta - 1). \quad (6.3)$$

Let N_0, Y_0, U_0 be the corresponding background black hole solution (4.4) and consider solutions that approach this background configuration at infinity, $N = N_0(1 + \delta N)$, $Y = Y_0(1 + \delta Y)$, $U = U_0(1 + \delta U)$ where $\delta N, \delta Y, \delta U$ vanish for $r \rightarrow \infty$. Inserting into (5.7) and

linearizing gives

$$\delta N = \frac{A \sin^2 \eta}{r^3} + O(\delta U), \quad \delta Y = O(\delta U), \quad \delta U = B_1 e^{\lambda_1 r} + B_2 e^{\lambda_2 r} \quad (6.4)$$

with

$$\lambda_1 = -2 + \sqrt{\frac{2 \mp 5 \cos \eta + \cos^2 \eta}{1 \mp \cos \eta}}, \quad \lambda_2 = -2 - \sqrt{\frac{2 \mp 5 \cos \eta + \cos^2 \eta}{1 \mp \cos \eta}}. \quad (6.5)$$

Since $\Re(\lambda_1) < 0$ (unless for $\eta = 0, \pi$) and $\Re(\lambda_2) < 0$, both the λ_1 and λ_2 modes are acceptable. The asymptotic solution therefore contains three integration constants A, B_1, B_2 , which is enough to fulfill the three matching conditions within the shooting method.

C. Deformations of the Schwarzschild-de Sitter black hole.

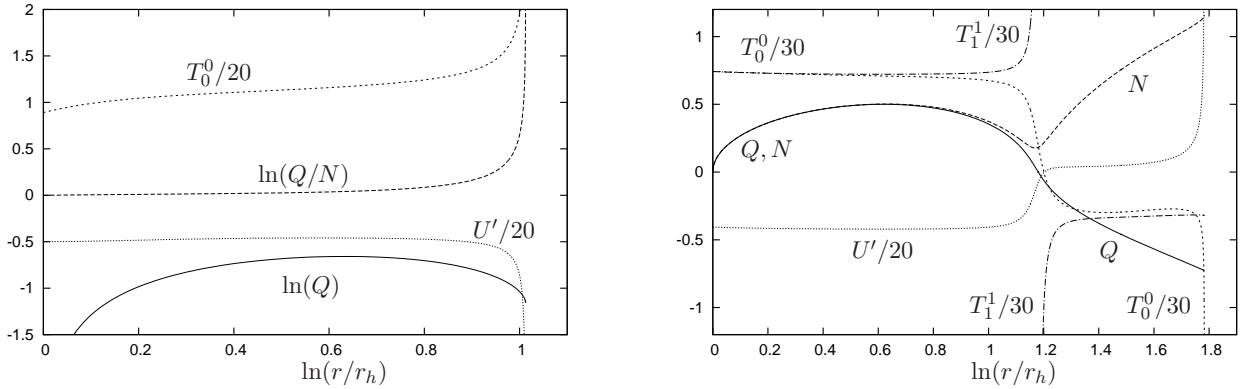


Figure 3. Deformations of the asymptotically dS solution for $u = -8$ (left) and $u = -9$ (right).

Let us now consider the last value in (4.6), $C = C_4 = -8.5566$, which gives rise to the SdS black hole (4.4) with $\Lambda = +21.5625 m^2$. Apart from the event horizon at $r = 1$, this solution has a cosmological horizon at $r = r_c$ where $N(r_c) = Q(r_c) = 0$. Let us set $u = C + \delta u$. Integrating the equations shows that the solution becomes singular at a finite distance from the horizon. If $\delta u < 0$ then the singularity is located at $r \approx r_c$ where N vanishes while Q does not, while the derivatives Q' , $(N^2)'$ and the energy density T_0^0 diverge (see Fig.3). This implies divergence of the Riemann tensor.

Somewhat peculiar features are shown by solutions with $\delta u > 0$. In this case Q develops a simple zero at $r \approx r_c$ but N remains finite, so that T_0^0 is also finite while T_r^r shows a simple pole. The curvature diverges at this point. Curiously, despite the latter, the solution can be further continued until a point where Q, N, T_r^r remain finite but U', N' diverge as does

the energy density T_0^0 which goes to *minus* infinity (see Fig.3). This produces a curvature singularity, as well as an infinite violation of the weak energy condition.

Summarizing, the Schwarzschild-de Sitter black hole does not admit non-compact, regular hairy generalizations, since all its deformations develop a curvature singularity at a finite proper distance from the horizon.

VII. GENERIC BLACK HOLES

So far we have been considering solutions parametrically close to the background black holes, that is for $u = C_k + \delta u$ where δu is not too large. Let us now consider what happens for arbitrary u , when we deviate further and further away from the background value $u = C_k$.

To begin with, solutions do not always exist, since for some values of u the argument of the square root in (5.9) may become negative thus rendering the parameter $a_1(u)$ and the local solution (5.8) complex-valued. Secondly, even if $a_1(u)$ in (5.9) is real, it should be positive, since we assume that N^2 grows at $r = r_h$ (black hole horizon). (We do not consider the case where N^2 decreases at $r = r_h$ (cosmological horizon), however it can be treated by making a formal replacement $N \rightarrow iN$, $Q \rightarrow i$, $Y \rightarrow iY$, $a \rightarrow ia$ in the field equations.)

Having determined the allowed values of u , we integrate the equations and find that the solutions always reproduce one of the types described above. They approach either one of the U, a backgrounds (4.7) so that $U'(r) \rightarrow 0$ as $r \rightarrow \infty$, or they are asymptotically AdS with the cosmological constant determined by (4.6) so that $U'(r) \rightarrow C_2$ or $U'(r) \rightarrow C_3$, or they are compact and singular.

We first study the upper branch solutions, with $a_1 = a_1^+(u)$ in (5.8), and find that they are non-compact only when u belongs to one of the following regions: $I_1^+ = [-0.53; -0.65]$, $I_2^+ = [1; 1.04]$, $I_3^+ = [8.3; 14.9]$ (see Fig.4). Solutions for $u \in I_1^+, I_2^+$ are the described above deformations of the background AdS black hole with $U'(r) = C_2$ and of the Schwarzschild solution. Solutions for $u \in I_3^+$ are new, for $r \rightarrow \infty$ they approach the background AdS black hole for $U'(r) = C_3$ but cannot be viewed as its continuous deformations, because they belong to the different branch and so the boundary conditions at the horizon are different.

For all other values of u the upper branch solutions develop a curvature singularity at a finite proper distance from the horizon, so that they are compact. In particular, the upper branch solutions seem to exist for all u large and negative, but they seem to be all singular.

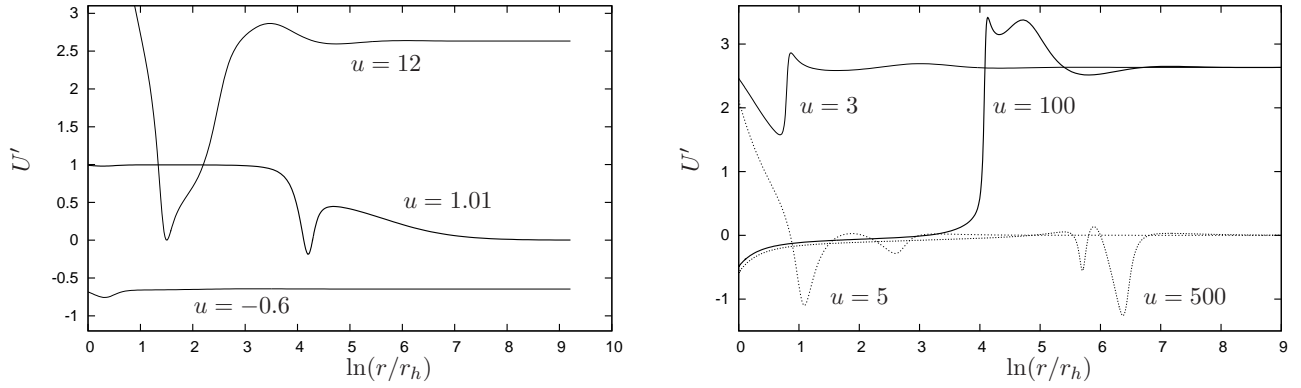


Figure 4. $U'(r)$ for the upper (left) and lower (right) branches of the generic non-compact solutions.

We then consider the lower branch solutions, with $a_1 = a_1^-(u)$ in (5.8), and find that they are non-compact if u belongs to one of the four regions: $I_1^- = [2.63; 3]$, $I_2^- = [3.1; 29.5]$, $I_3^- = [29.9; 154.2]$, $I_4^- = [154.4; 1876]$. Solutions for $u \in I_1^-$ are the deformations of the background black hole with $U'(r) = C_3$. For $u \in I_2^-$ and $u \in I_4^-$ we find new asymptotically U, a solutions, not continuously related to the Schwarzschild metric, because they belong to the different branch and so have different boundary conditions at the horizon. For $u \in I_3^-$ we obtain new solutions that approach the background AdS black hole with $U'(r) = C_3$. It seems that for all other values of u the solutions are compact and singular.

When a solution from one branch is regular and non-compact, the one for the same u from the second branch is usually compact and singular. However, this is not a general rule, since, for example, the intervals I_3^+ and I_2^- have a non-zero overlap.

The described above picture corresponds to the theory parameters $m = 0.1$, $\eta = 1$, $c_3 = 0.1$, $c_4 = 0.3$. Varying these values changes the size, position and number of intervals of u within which non-compact solutions exist. However, the overall picture remains the same: the solutions either approach asymptotically the U, a backgrounds (4.7) so that $U'(r) \rightarrow 0$ as $r \rightarrow \infty$, or they are asymptotically AdS with $U'(r) \rightarrow C$ where C is a root of (4.6) such that $\Lambda(C) < 0$, or they are compact and singular. If $4c_3 + c_4 > 6$ then the cosmological term in (4.7) changes sign and the solutions that used to be asymptotically U, a for $4c_3 + c_4 < 6$ become compact and singular.

The only asymptotically flat solution we find for generic parameter values is the pure Schwarzschild black hole. Similarly, the only asymptotically dS solution is the pure SdS.

VIII. SPECIAL SOLUTIONS

It is possible that new solutions could exist for special parameter values. Let us see if they could be asymptotically flat. We know that the cosmological constant $\Lambda(C)$ in (4.5) vanishes for $C = 1$. However, it will also vanish if the expression in the parenthesis in (4.5) vanishes, that is for

$$C = C_{\pm} = \frac{1}{2(c_3 + c_4)} \left(2c_4 + 5c_3 - 3 \pm \sqrt{12c_4 + 9(c_3 - 1)^2} \right). \quad (8.1)$$

Inserting this to (4.3), the polynomial $P(C)$ will vanish if either $c_4 = -(2/3)c_3^2$ or if $\eta = 0$, in which cases we obtain additional background black holes described by the Schwarzschild metric. Let us now see what happens if we deform these solutions by setting at the horizon $u = C_{\pm} + \delta u$.

It turns out that nothing very interesting happens in the case where η is arbitrary and $c_4 = -(2/3)c_3^2$. There are two Schwarzschild solutions in this case, one for $u = C_+$ and another one for $u = C_-$. Choosing $u = C_{\pm} + \delta u$ produces a curvature singularity at a finite proper distance from the horizon, similar to what is shown in Fig.3.

Let us now consider the case where c_3, c_4 are arbitrary but $\eta = 0$. When η vanishes, the source term in equation (2.7) vanishes too, so that the metric $f_{\mu\nu}$ becomes Ricci-flat. However, it cannot be flat, since our boundary conditions require that it should have a horizon, so that it becomes the vacuum Schwarzschild metric. Therefore, unless we change the boundary conditions, we do not recover for $\eta \rightarrow 0$ the RGT theory where $f_{\mu\nu}$ is non-dynamical and flat, but obtain instead the theory where $f_{\mu\nu}$ is non-dynamical and Schwarzschild. The line element (3.1) then can be represented in the form

$$f_{\mu\nu}dx^{\mu}dx^{\nu} = a^2(U)dt^2 - \frac{dU^2}{Y^2(U)} - U^2(d\vartheta^2 + \sin^2\vartheta d\varphi^2) \quad (8.2)$$

where

$$Y(U) = \sqrt{1 - \frac{u}{U}}, \quad a(U) = AY(U) \quad (8.3)$$

with A being an integration constant. It is easy to check that this choice of $Y(U), a(U)$ solves equations (3.6c) and (3.6d).

Now, setting $u = C_{\pm}$ and integrating equations (5.7) we find that the metric $g_{\mu\nu}$ is also Schwarzschild as it should be. Setting $u = C_- + \delta u$ gives solutions which are close to the Schwarzschild metric in the event horizon vicinity but develop a curvature singularity at a finite distance from the horizon, so that this case is also not very interesting.

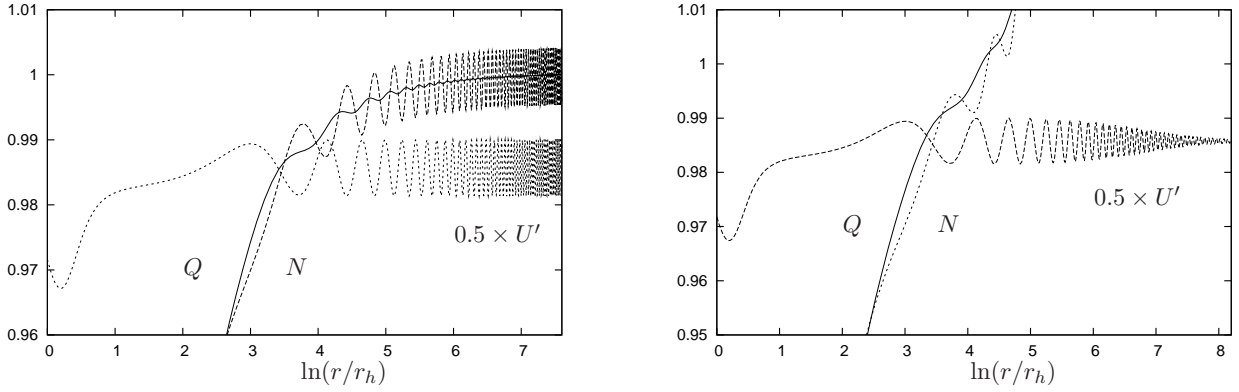


Figure 5. Solutions with $c_3 = 0.1$, $c_4 = 0.3$, $u = 1.99$ and $\eta = 0$ (left) and $\eta = 0.02$ (right).

Qualitatively new solutions arise for $u = C_+ + \delta u$. For example, for $c_3 = 0.1$ and $c_4 = 0.3$ one has $C_+ = 1.97133$, while setting $u = 1.99$ gives the result shown in Fig.5. These solutions stay always close to the background Schwarzschild solution but do not tend to it for $r \rightarrow \infty$ and show instead infinitely many oscillations with a constant amplitude. One has

$$N = \sqrt{1 - \frac{1}{x}} + \delta N, \quad Q = \sqrt{1 - \frac{1}{x}} + \delta Q, \quad U = C_+ x + \delta U, \quad (8.4)$$

where δN , δQ , δU are small everywhere. Linearizing the field equations gives for large r

$$\delta U = \exp\{i\sqrt{2}m(x + \frac{1}{2}\ln(x))\} + \dots, \quad \delta N = -\frac{im}{\sqrt{2}}\delta U + \dots, \quad \delta Q = \frac{1}{x}\delta U + \dots, \quad (8.5)$$

where the real parts should be taken, the dots stand for the subleading terms, and where we have assumed for simplicity that $c_3 = c_4 = 0$, in which case $C_+ = 2$. We see that the solutions behave as if the graviton mass was imaginary, thus providing a tachyonic version of the asymptotic behavior (6.1),(6.2) obtained by linearizing around the $C = 1$ Schwarzschild black hole. It is interesting that the oscillations persist even for finite (but small) values of η , but the solutions become then asymptotically AdS and the oscillation amplitude decreases with r as shown in Fig.5.

IX. GLOBALLY REGULAR SOLUTIONS – STARS AND LUMPS

For the sake of completeness, let us briefly study what happens when there is no horizon and the solutions are globally regular. They can couple to a compact matter source, in which case they turn out to be asymptotically flat. If the source is absent, then the solutions describe globally regular ‘lumps of self-gravitating energy’.

A. Stars

Let us return to equations (3.6a)–(3.6e) and restore non-zero values of ρ, P . We assume the energy density to be constant inside the star and to vanish outside, so that $\rho(r) = \rho_\star$ if $r < R_\star$ and $\rho(r) = 0$ for $r > R_\star$. The pressure $P(r)$ is determined by equation (3.5) which should be solved simultaneously with the other equations. The pressure should vanish at the surface of the star, where the metrics $g_{\mu\nu}$ and $f_{\mu\nu}$ should be continuous. We require both metrics to be regular at the origin $r = 0$, where the curvature should be finite. This leads to the local power-series solution at small r (assuming for simplicity that $c_3 = c_4 = 0$)

$$\begin{aligned} N &= 1 + \left(m^2 \cos^2 \eta \left(1 - \frac{3}{2}u + \frac{1}{2}u^2 \right) - \frac{\rho_\star}{6} \right) r^2 + O(r^4), & U &= ur + O(r^3), \\ Y &= 1 + m^2 \sin^2 \eta \frac{u-1}{2u} x^2 + O(r^4), & P &= p + O(r^2), & Q &= q + O(r^2), \\ a &= a(0) + O(r^2), \end{aligned} \tag{9.6}$$

where u, p, q are free parameters which determine the value of $a(0)$. We now wish to extend these local solutions numerically towards large r in order to match the flat asymptotics (6.1),(6.2). Let us again count the free parameters. We need to integrate the 5 first order equations – these are Eqs.(5.7),(5.3c) (generalized to the case of non-zero ρ, P) and also equation (3.5) for the pressure. In order to get the solutions within the multiple shooting method, we need 5 free parameters in order to match the values of 5 amplitudes N, Y, U, Q, P at the matching point (the amplitude a is obtained afterwards from the algebraic constraint (5.2)). And indeed, we have in our disposal exactly 5 free parameters – these are u, q, p in (9.6) and A, B in the asymptotic solution (6.1),(6.2). We can therefore do the matching, and this gives us the global solutions. It is convenient to introduce mass functions M_g, M_f defined by $N^2 = 1 - 2M_g/r$ and $Y^2 = 1 - 2M_f/U$, in terms of which the 00-components of the Einstein equations (3.2) read

$$(M_g)' = \frac{r^2}{2} (m^2 \cos^2 \eta T_0^0 + \rho), \tag{9.7a}$$

$$(M_f)' = U' \frac{U^2}{2} m^2 \sin^2 \eta \mathcal{T}_0^0. \tag{9.7b}$$

The typical solutions are shown in Fig.6. We see that inside the star, for $r < R_\star$, the pressure falls from its maximal value at the center till zero at the star surface, while the mass function M_g rapidly increases. The mass function M_f also increases but not as fast,

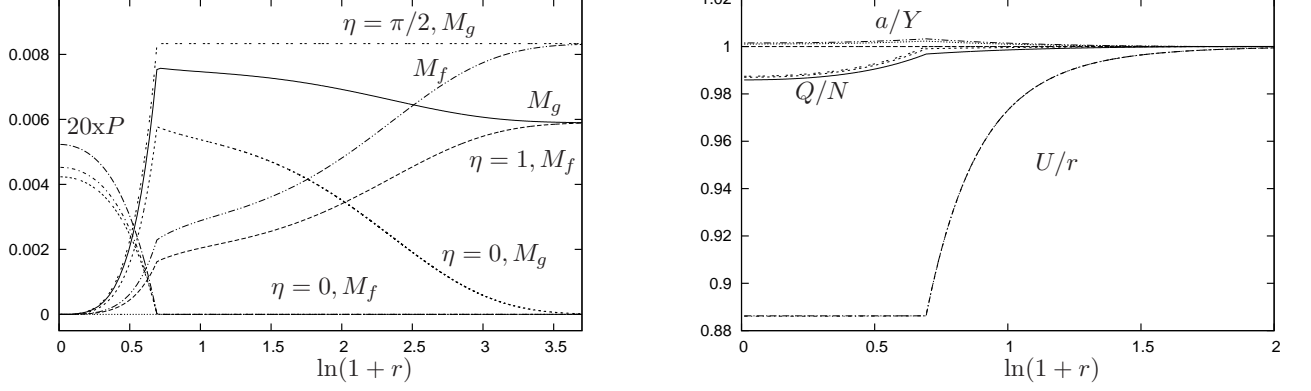


Figure 6. Star solutions for $c_3 = c_4 = 0$, $m = 0.2$, $\rho_\star = 0.05$, $R_\star = 1$ for three different values of η . The surface of the star is located where P vanishes. The behavior of $U(r)$ practically does not change when η changes, so that the three curves for U/r almost coincide and look like as one curve.

since it does not couple directly to ρ but only to \mathcal{T}_0^0 . Outside the star M_g decreases (for $\eta < \pi/2$) because it is then sourced only by T_0^0 which is negative, while M_f still increases, since \mathcal{T}_0^0 is positive. For large r both mass functions approach the same asymptotic value $M_g(\infty) = M_f(\infty) = A \sin^2 \eta$ required by (6.1). When varying η the parameter A almost does not change, so that $A \sin^2 \eta$ changes from the maximal value for $\eta = \pi/2$ to zero for $\eta = 0$.

For $\eta = \pi/2$ the metric $g_{\mu\nu}$ decouples and is described by the Schwarzschild solution for the perfect fluid. One has in this case

$$M_g = \rho_\star \frac{r^3}{6} \quad \text{for } r < R_\star \quad \text{and} \quad M_g = M_{\text{ADM}} = \rho_\star \frac{R_\star^3}{6} \quad \text{for } r > R_\star \quad (9.8)$$

so that the mass function M_g varies only inside the star, while outside it is constant and equals to its asymptotic value – the ADM mass of the spacetime. The mass function M_f grows monotonically till the value M_{ADM} .

If $\eta < \pi/2$ then M_g grows inside the star but not as fast as for $\eta = \pi/2$, since the positive contribution of ρ to the right hand side of (9.7a) is partially screened by the negative T_0^0 . Outside the star T_0^0 remains negative and continues to screen the mass of the star, so that M_g approaches at infinity a lesser value than it has at the surface of the star.

For $\eta = 0$ the metric $f_{\mu\nu}$ decouples and becomes flat, so that $a = Y = 1$, $M_f = 0$, and we recover the RGT theory (unlike for the black holes). The positive mass of the star is then completely screened by the negative graviton energy, so that the mass function M_g

approaches zero at infinity. This follows from the fact that the $1/r$ terms in the metric should be absent, since the metric should now approach its asymptotic value exponentially fast.

These solutions show the Vainshtein mechanism of recovery of General Relativity [8]. Indeed, when the graviton mass m is very small, the contribution of $m^2 T_0^0$ to the total energy density in Eq.(9.7a) becomes small as compared to ρ . The mass function M_g then stays approximately constant in a large region outside the star, in which case General Relativity is a good approximation.

B. Lumps of pure gravity

Let us now return to the local solution at the origin (9.6) and set $\rho_\star = p = 0$, in which case there is no source, and there is essentially only one free parameter left, u . Integrating towards large r , we obtain a family of solutions labeled by u and describing lumps of pure gravity. They have a regular center at $r = 0$, while for large r they behave similarly to the black holes and approach either the AdS backgrounds (4.4), or the U, a backgrounds (4.7), or develop a curvature singularity at a finite proper distance from the center. It seems

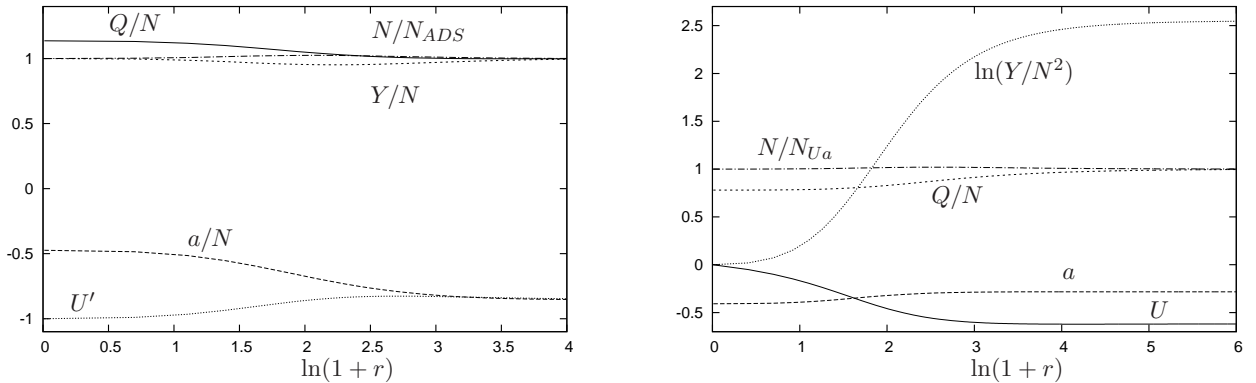


Figure 7. Lump for $c_3 = c_4 = 0$, $m = 0.2$, $\eta = 1$ and for $u = -1$ (left) and $u = -0.1$ (right).

that the lumps are related to the black holes and can be obtained from them in the limit when the event horizon shrinks. Such a phenomenon is actually well known for hairy black holes, which can often be viewed as non-linear superpositions of a regular gravitating matter configuration (solution) with a vacuum black hole [26].

The lump shown in the left panel of Fig.7 approaches for $r \rightarrow \infty$ the AdS background

(4.4) with $C = -0.8508$, while the one on the right tends to the U, a background (4.7). These solutions can be viewed as regular deformations of the background and can be approximately described by replacing the constant M in (4.4),(4.7) by a function that approaches a constant value at large r but vanishes at $r = 0$, thus insuring the regularity at the origin.

X. CONCLUDING REMARKS

There is something peculiar about black holes in massive gravity – all known solutions are of the Schwarzschild(anti)-de Sitter type [22], [13], [14], [15], and it is unclear if there could be others, asymptotically flat, say. New black holes could exist if only the two metrics are simultaneously diagonal, but since their event horizons should then coincide, this excludes the bimetric theories from consideration [24]. However, the argument does not apply in the bigravity theories, so that new black holes could exist there. Their quest was the subject of the above analysis.

We did indeed find a whole zoo of new black holes with massive degrees of freedom excited. All of them have a regular event horizon common for both metrics, while in the outside region they show various types of behavior. Some of them look more conventional and support a short massive hair around the horizon. Some of them exhibit peculiar properties, as for example infinite and negative energy T_0^0 (negative energies seem to be common for generic solutions), or the tachyonic features. However, none of them appear to be asymptotically flat. It seems therefore that asymptotic flatness is incompatible with an event horizon as soon as the massive degrees of freedom are excited. This could perhaps be interpreted in the spirit of the no-hair theorems, many of which state that asymptotically flat black holes cannot support massive hair (see [26] for a review and references).

We have also constructed regular vacuum solutions without a horizon – lumps of pure gravity. However, they are also not asymptotically flat and show in the far field the same behavior as the black holes. It seems that they can be obtained from the latter in the limit when the horizon shrinks to zero.

However, adding a matter source we do find asymptotically flat solutions which exhibit the Vainshtein mechanism of recovery of General Relativity at finite distances. It seems therefore that the mechanism needs a matter source and does not work for pure vacuum systems like black holes. This can be supported by a qualitative argument, quite in the spirit

of Vainstein's original argumentation [8]. If there is a matter source, then schematically one has

$$G_{\lambda}^{\rho} = m^2 T_{\lambda}^{\rho} + T_{\lambda}^{(\text{m})\rho}, \quad (10.1)$$

and so it is clear that if the graviton mass m is very small, then one should be able in some way to neglect the first term on the right as compared to the second one. This reproduces General Relativity. However, if $T_{\lambda}^{(\text{m})\rho} = 0$, as for the studied above black holes or lumps, then there is no justification for omitting $m^2 T_{\lambda}^{\rho}$, and General Relativity is not recovered.

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APPENDIX. Solutions with decoupled metrics.

In this Appendix we analyze the possibility to have a non-vanishing coefficient c in the metric $f_{\mu\nu}$ in (2.15). This metric is then non-diagonal, which allows to find new solutions. However, these solutions are essentially of the same type as those found long ago by Isham and Storey [22], in particular they reduce to those found in [13],[14] when $\eta \rightarrow 0$.

The expression τ_r^0 in (2.18) must vanish, which is possible for $c \neq 0$ if only the expression in the parentheses vanishes. The latter condition requires that U must be proportional to r , $U = Cr$, in which case

$$\tau_r^0 = \frac{c}{Q} \{2C - 3 + c_3(C^2 - 4C + 3) + c_4(C - 1)^2\}, \quad (A.1)$$

which can be set to zero by adjusting the value of C . Equivalently, one can consider C, c_3 as independent parameters, which will be assumed below, and this implies that

$$c_4 = -\frac{2C - 3 + c_3(C^2 - 4C + 3)}{(C - 1)^2}. \quad (A.2)$$

This also implies that $T_0^0 = T_r^r = \lambda$ with λ given by Eq.(A.5) below, in which case the conservation condition (3.3) becomes

$$\stackrel{(g)}{\nabla}_{\mu} T_r^{\mu} = \frac{2}{r}(T_r^r - T_{\vartheta}^{\vartheta}) = \frac{2(c_3 C - C - c_3 + 2)(C^2 Q - C Q N b - a C + c^2 N^2 Q + a b N)}{r(C - 1)Q} = 0. \quad (A.3)$$

Assuming for a time being that this is true, we shall solve the equations and later shall impose this condition on the solutions. Provided that (A.3) is true, the energy-momentum tensors assume a very simple form

$$T_\nu^\mu = \lambda \delta_\nu^\mu, \quad \mathcal{T}_\nu^\mu = \tilde{\lambda} \delta_\nu^\mu, \quad (\text{A.4})$$

with

$$\lambda = (C - 1)(c_3 C - C - c_3 + 3), \quad \tilde{\lambda} = \frac{1 - C}{C^2} (c_3 C - c_3 + 2), \quad (\text{A.5})$$

so that the field equations (2.6),(2.7) reduce to

$$G_\nu^\mu = m^2 \cos^2 \eta \lambda \delta_\nu^\mu, \quad (\text{A.6})$$

$$\mathcal{G}_\nu^\mu = m^2 \sin^2 \eta \tilde{\lambda} \delta_\nu^\mu. \quad (\text{A.7})$$

These equations describe the dynamics of the two metrics independently driven by their cosmological terms. The equations for $g_{\mu\nu}$ completely decouple from those for $f_{\mu\nu}$ so that we can solve them independently. With $g_{\mu\nu}$ given by (2.17), the solution of Eqs.(A.6) is the Schwarzschild-(anti)de Sitter metric,

$$Q^2 = N^2 = 1 - \frac{2M}{r} - \frac{\lambda}{3} m^2 \cos^2 \eta r^2. \quad (\text{A.8})$$

Let us now consider equations (A.7) for $f_{\mu\nu}$. They are slightly more difficult to solve, since $f_{\mu\nu}$ is non-diagonal, while its components $f_{\vartheta\vartheta} = U^2$ and $f_{\varphi\varphi} = U^2 \sin^2 \vartheta$ are already fixed, since $U = Cr$. However, the components f_{00} , f_{0r} , f_{rr} are still free, because they contain three up to now unspecified functions a, b, c . We can consider U as the new radial coordinate, changing at the same time the temporal coordinate, so that $t \rightarrow T(t, r)$, $r \rightarrow U = Cr$. The metric then becomes

$$f_{\mu\nu} dx^\mu dx^\nu = f_{TT} dT^2 + 2f_{TU} dT dU + f_{UU} dU^2 - U^2 (d\vartheta^2 + \sin^2 \vartheta d\varphi^2), \quad (\text{A.9})$$

where f_{TT}, f_{TU}, f_{UU} are functions of T, U . The structure of the source term in (A.7) remains the same in the new coordinates, so that we should solve the Einstein equations with the cosmological term to find a metric parameterized by the radial Schwarzschild coordinate U . The solution is the (anti)de Sitter metric

$$f_{\mu\nu} dx^\mu dx^\nu = \Delta dT^2 - \frac{dU^2}{\Delta} - U^2 (d\vartheta^2 + \sin^2 \vartheta d\varphi^2), \quad (\text{A.10})$$

where $\Delta(U) = 1 - \frac{\tilde{\lambda}}{3} m^2 \sin^2 \eta U^2$. There remains to establish the correspondence between the T, U and t, r coordinates. Let us introduce 1-forms

$$\theta^0 = \sqrt{\Delta} dT, \quad \theta^1 = \frac{dU}{\sqrt{\Delta}}, \quad \theta^2 = U d\vartheta, \quad \theta^3 = U \sin \vartheta d\varphi, \quad (\text{A.11})$$

such that $f_{\mu\nu} = \eta_{AB} \theta_\mu^A \theta_\nu^B$. The correspondence between the T, U and t, r coordinates can be established by relating the 1-forms θ_μ^A from (A.11) to ω_μ^A from (2.13). The two sets of 1-forms need not coincide but may differ by a local Lorentz rotation. This gives two conditions

$$\omega^0 = \sqrt{1 + \alpha^2} \theta^0 + \alpha \theta^1, \quad \omega^1 = \sqrt{1 + \alpha^2} \theta^1 + \alpha \theta^0, \quad (\text{A.12})$$

where α is the rotation parameter. These conditions explicitly read

$$\begin{aligned} a dt + c dr &= \sqrt{1 + \alpha^2} (\sqrt{\Delta} \dot{T} dt + \sqrt{\Delta} T' dr) + \alpha \frac{C dr}{\sqrt{\Delta}}, \\ -c N^2 dt + b dr &= \sqrt{1 + \alpha^2} \frac{C dr}{\sqrt{\Delta}} + \alpha (\sqrt{\Delta} \dot{T} dt + \sqrt{\Delta} T' dr). \end{aligned} \quad (\text{A.13})$$

Comparing the coefficients in front of dt, dr one finds

$$T(t, r) = Ct + C \int f(r) dr \quad (\text{A.14})$$

and

$$a = C \sqrt{1 + \alpha^2} \sqrt{\Delta}, \quad b = \sqrt{1 + \alpha^2} \frac{C}{\sqrt{\Delta}} + C \alpha \sqrt{\Delta} f, \quad c = -C \alpha \frac{\sqrt{\Delta}}{N^2} \quad (\text{A.15})$$

with

$$f = -\frac{\alpha}{\sqrt{1 + \alpha^2}} \frac{N^2 + \Delta}{N^2 \Delta}. \quad (\text{A.16})$$

In order to determine the yet unspecified function α , we now use the condition that the expression in (A.3) has to vanish. It will vanish if (since $Q = N$)

$$C^2 N - C N^2 b - a C + c^2 N^3 + a b N = 0, \quad (\text{A.17})$$

which will be satisfied if

$$\alpha = \frac{N^2 - \Delta}{2N\sqrt{\Delta}}. \quad (\text{A.18})$$

Together with $U = Cr$ this finally establishes the correspondence between the t, r and T, U coordinates and specifies all components of $f_{\mu\nu}$.

The above considerations give a family of the Schwarzschild-(anti)de Sitter backgrounds in the bigravity theory for generic values of the parameters. Some additional care should

be taken in order to make sure that all coefficients in the above expressions are real. For example, if the parameters are chosen such that $l > 0$ and $\tilde{l} < 0$, then $\sqrt{\Delta}$ will be always real, while N will be real in the region between the black hole and cosmological horizons. The solutions in regions beyond the horizons can be obtained by the analytic continuation.

When $\eta \rightarrow 0$ then the metric $f_{\mu\nu}$ becomes flat while the metric $g_{\mu\nu}$ does not change. The solutions then reduce to those obtained in the RGT theory [13].

Another possibility to set to zero the expression in (A.3) is to restrict the value of the coefficient c_3 in such a way that the first factor in the numerator in (A.3) vanishes,

$$c_3 C - C - c_3 + 2 = 0, \quad (\text{A.19})$$

which implies that

$$c_3 = \frac{C-2}{C-1}, \quad c_4 = -\frac{C^2-3C+3}{(C-1)^2}, \quad l = C-1, \quad \tilde{l} = \frac{1-C}{C}. \quad (\text{A.20})$$

Since one does not need to impose the condition (A.17) in this case, there is no equation for the parameter α in (A.14),(A.15),(A.16) so that it remains arbitrary. Choosing $\alpha = 0$, the $\eta \rightarrow 0$ limit of such solutions corresponds to the case considered in Ref.[14].

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